PMM U.S.S.R.,Vol.44,pp.180-182 0021-8928/81/2 0180 $7.50/0

UDC 62-50

UPPER ESTIMATE FOR THE VALUE OF A LINEAR DIFFERENTIAL GAME *

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A fixed-duration differential game with linear dynamics and convex terminal payoff is examined. An upper estimate is given for the game's value, bases on an analysis of functions of a sequential program maximin. A number of papers exist (see /1,2/) on the construction of monotone approximations to the game's value. The question arises of estimating the error. For a fixed partitioning of the time interval of the game a sequential program maximin /3/ of the position being examined can be computed; it is an approximation from below to the game's value. An upper estimate for the value is given in the present paper, and, in this way, an error estimate is obtained.

Let a system's motion be described by the equation

$$x^{\cdot} = u + v,\ x(t_0) = x_0,\ x \in R^n,\ t \in [t_0, \theta],\ u \in P(t),\ v \in Q(t) \tag{1}$$

Here R^n is an n-dimensional Euclidean space, $P(t)$ and $Q(t)$ are convex compacta in R^n, depending continuously on t, bounding the control vectors of the first and second players, respectively. The first (second) player strives to minimize (maximize) the payoff $g(x(\theta))$, viz., the value of function $g(x)$ on the system's phase vector at the final instant θ. Function g is assumed to be convex and to satisfy a Lipschitz condition with constant α. Also, $g(x) \to +\infty$ as $x \to \infty$. This is not a limitation. The game is analyzed in the position formalization /3/. For every position (t, x) there exists a game value $\varepsilon(t, x)$ /3/. We shall estimate the quantity $\varepsilon(t_0, x_0)$.

Let $\Delta = \{t_0 < t_1 < \ldots < t_N = \theta\}$ be a partitioning of interval $[t_0, \theta]$. By $\varepsilon_\Delta{}^i(x)$ $(i = N, N-1, \ldots, 0)$ we denote the value of a sequential program maximin corresponding to partitioning Δ and computed for the position (t_i, x), i.e.

$$\varepsilon_\Delta^N(x) = g(x), \quad \varepsilon_\Delta^i(x) = \max_{v(\cdot)} \min_{u(\cdot)} \varepsilon_\Delta^{i+1}\Big(x + \int_{t_i}^{t_{i+1}} (u(t) + v(t))\, dt\Big)$$

Here $u(t) \in P(t)$ and $v(t) \in Q(t)$ are measurable functions on $[t_i, t_{i+1}]$. We remark that the functions $\varepsilon_\Delta{}^i$, convex in x, satisfy the Lipschitz condition with constant α. It is well known /3/ that the lower estimate

$$\varepsilon_\Delta{}^\circ(x_0) \leqslant \varepsilon(t_0, x_0) \tag{2}$$

is valid for the value $\varepsilon(t_0, x_0)$ and that $\varepsilon_\Delta{}^\circ(x_0) \to \varepsilon(t_0, x_0)$ when the diameter of partitioning $\Delta \to 0$. To determine the error it is enough to estimate $\varepsilon(t_0, x_0)$ from above.

For each fixed $i = 0, 1, \ldots, N-1$ we introduce a function $d_i(\omega)$ $(\omega \geqslant \min_x \varepsilon_\Delta{}^i(x)$; this minimum exists since the continuous function $\varepsilon_\Delta{}^i(x) \to +\infty$ as $\|x\| \to \infty)$. For every such ω let $\rho_\omega{}^i(l)$ be the support function /4/ of set $\{y \in R^n\colon \varepsilon_\Delta{}^{i+1}(y) \leqslant \omega\}$ and let

$$f_\omega{}^i(t, x) = \max_{\|l\| \leqslant 1} \Big\{\langle l, x\rangle + \int_t^{t_{i+1}} H(\tau, l)\, d\tau - \rho_\omega{}^i(l)\Big\} \tag{3}$$

$$H(\tau, l) = \min_{u \in P(\tau)} \langle l, u\rangle + \max_{v \in Q(\tau)} \langle l, v\rangle, \quad l \in R^n, \quad \tau \in [t_0, \theta]$$

(the maximum in (3) exists since function $\rho_\omega{}^i$ is finite and continuous). Let $I_i(\omega) = [t_i, t_i(\omega)] \subset [t_i, t_{i+1}]$ be some interval (a point can be taken as an interval) such that for some $\delta > 0$ and for every position (t_*, x_*) from the domain

$$\{(t, x)\colon t \in (t_i, t_i(\omega)),\ 0 < f_\omega{}^i(t, x) < \delta\}$$

there is fulfilled the

Condition (see Condition 43.2 in /3/). For every $v_* \in Q(t_*)$ there exists $u_* \in P(t_*)$ such that the inequality

$$\langle l, u_*\rangle + \langle l, v_*\rangle \leqslant H(t_*, l)$$

is valid for all vectors l on which the maximum in (3) is achieved when $t = t_*$ and $x = x_*$.

We observe that we can always take $I_i(\omega) = [t_i, t_i]$. Let

$$c_i(\omega) = \alpha \min_{t \in I_i(\omega)} \int_t^{t_{i+1}} \beta_i(\tau, t)\, d\tau, \qquad (t \leqslant \tau \leqslant t_{i+1})$$

*Prikl Matem. Mekhan., 44, No.2, 254-257, 1980

$$\beta_i(\tau, t) = \max\left\{0, \max_{\|l\|=1}(H(\tau, l) - \frac{1}{t_{i+1}-t}\int_t^{t_{i+1}} H(s, l)ds)\right\}$$

We define the function d_i by the formula $d_i(\omega) = \inf_{r\geqslant\omega}(r + c_i(r))$.

Theorem. The inequality

$$\varepsilon(t_0, x_0) \leqslant d_{N-1}(d_{N-2}\ldots(d_0(\varepsilon_\Delta{}^\circ(x_0)))\ldots)$$

is valid.

To prove this we need the following

Lemma. Let $N = 1$ and let $\varepsilon_\Delta{}^\circ(x_0) = \varepsilon^\circ(t_0, x_0)$ be the usual program maximin of position (t_0, x_0). Then

$$\varepsilon(t_0, x_0) - \varepsilon^\circ(t_0, x_0) \leqslant \alpha\int_{t_0}^{\theta}\beta_0(\tau, t_0)\,d\tau$$

Let us present the Lemma's proof. We consider the game obtained from the original one by replacing sets $P(t)$ and $Q(t)$ by the t-independent compacta

$$P_* = \frac{1}{\theta - t_0}\int_{t_0}^{\theta} P(t)\,dt, \quad Q_* = \frac{1}{\theta - t_0}\int_{t_0}^{\theta} Q(t)\,dt$$

Then the program maximins for position (t_0, x_0) for the new and the original games coincide, and, by virtue of /5/, the value coincides with the program maximin for a linear game with a simple motion (i.e., when the sets bounding the controls do not depend on t) and with a convex payoff. Thus, to obtain Lemma's assertion, it is necessary to compare values of initial and construction games. This is done by using unification Theorem /6/.

Proof of the Theorem. Let us show that

$$\varepsilon(t_{N-1}, x_*) \leqslant d_{N-1}(\varepsilon_\Delta^{N-1}(x_*)), \quad \forall x_* \in R^n \tag{4}$$

We take an arbitrary number $r \geqslant \varepsilon_\Delta{}^{N-1}(x_*)$ and we consider the set

$$\{(t, x): t \in I_{N-1}(r), f_r^{N-1}(t, x) = 0\} \tag{5}$$

This set is formed of all positions (t, x), $t \in I_{N-1}(r)$, from which a program absorbtion /3/ of set $\{y: \varepsilon_\Delta{}^N(y) \leqslant r\}$ at instant t_N is possible. Taking the definition of interval $I_i(\omega)$ into account, from /3/ we have that set (5) is u-stable and, consequently /3/, the first player can ensure that every motion starting off from position (t_{N-1}, x_*) remains in this set as long as $t \in I_{N-1}(r)$. But the inequality

$$\varepsilon(t, x) \leqslant r + \alpha\int_t^{\theta}\beta_{N-1}(\tau, t)\,d\tau \tag{6}$$

is valid for every position (t, x) from set (5); we obtain this inequality by applying the Lemma to the interval $[t, \theta]$ (replacing (t_0, x_0) by (t, x)) and taking into account that $\varepsilon^\circ(t, x) \leqslant r$ since a program absorbtion of set $\{y: g(y) \leqslant r\}$ is possible from (t, x). From what has been said above and from inequality (6) follows

$$\varepsilon(t_{N-1}, x_*) \leqslant r + c_{N-1}(r), \quad \forall r \geqslant \varepsilon_\Delta^{N-1}(x_*)$$

which proves the validity of (4).

Let us now prove that

$$\varepsilon(t_{N-2}, x_*) \leqslant d_{N-1}(d_{N-2}(\varepsilon_\Delta^{N-2}(x_*))), \quad \forall x_* \in R^n \tag{7}$$

Let ω_* be the value of position (t_{N-2}, x_*) in a game with final instant t_{N-1}, payoff $\varepsilon_\Delta{}^{N-1}(\cdot)$ and equation (1). Allowing for the fact that d_{N-1} is a nondecreasing function, from inequality (4) we obtain

$$\varepsilon(t_{N-2}, x_*) \leqslant d_{N-1}(\omega_*) \tag{8}$$

We take an arbitrary number $r \geqslant \varepsilon_\Delta{}^{N-2}(x_*)$. Since set

$$\{(t, x): t \in I_{N-2}(r), \quad f_r^{N-2}(t, x) = 0\}$$

is u-stable, the first player can ensure that every motion starting off from position (t_{N-2}, x_*) remains in this set as long as $t \in I_{N-2}(r)$. Whence, applying the Lemma to the intervals $[t, t_{N-1}]$ $(t \in I_{N-2}(r))$ and taking the function $\varepsilon_\Delta{}^{N-1}(\cdot)$ as the payoff, we get that

$$\omega_* \leqslant r + \alpha\int_t^{t_{N-1}}\beta_{N-2}(\tau, t)\,d\tau, \quad \forall t \in I_{N-2}(r)$$

Consequently,

$$\omega_* \leqslant d_{N-2}(\varepsilon_\Delta^{N-2}(x_*)) \tag{9}$$

Since function d_{N-1} is nondecreasing, (7) follows from (8) and (9). The theorem's assertion is proved by continuing analogously. Inequality (2) and the Theorem yield a two-sided estimate for the game's value.

Notes. 1°. The wider we are able to choose sets $I_i(\omega)$ the more exact is the estimate obtained. The crudest estimate is obtained if we take $I_i(\omega) = [t_i, t_i]$. Then

$$\varepsilon(t_0, x_0) \leqslant \varepsilon_\Delta{}^\circ(x_0) + a \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \beta_i(\tau, t_i)\, d\tau$$

We remark that if the diameter of the partitioning $\Delta \to 0$ then the estimate given by the theorem tends to $\varepsilon(t_0, x_0)$.

2°. Functions d_i are nondecreasing; therefore, for obtaining the upper estimate we can compute them not exactly but with an excess. To illustrate the fact that the error estimate is not excessive we consider a well-known example (see /3/). Let the game be

$$x^\cdot = u + v, \quad x \in R^2, \quad t \in [0, 1], \quad g(x) = \|x\|$$
$$P(t) = \{u: \ \|u\| \leqslant 2(1-t)\}, \quad Q(t) = \{v: \ \|v\| \leqslant 1\}$$

Let $\Delta = \{0, 1\}$ i.e., $N = 1$. Then $\varepsilon_\Delta{}^\circ(x)$ is simply the program maximin for position $(0, x)$ and $\varepsilon_\Delta{}^\circ(x) = \|x\|$. Here we can set $I_0(\omega) = [0, 1]$ if $\omega \geqslant 1/4$ and $I_0(\omega) = [0, 1/2 - \sqrt{1/4 - \omega}]$ if $\omega < 1/4$. Hence, $d_0(\omega) = \omega\ (= 1/4)$ if $\omega \geqslant 1/4\ (< 1/4)$. The Theorem's assertion gives that $\varepsilon(0, x) \leqslant \max(1/4, \varepsilon_\Delta{}^\circ(x) = \|x\|)$. We take the partitioning $\Delta_1 = \{0, 1/2, 1\}$. Then $\varepsilon(0, x) \geqslant \varepsilon_{\Delta_1}{}^\circ(x) = \max(1/4, \|x\|)$. Thus, $\varepsilon(0, x) = d_0(\varepsilon_\Delta{}^\circ(x))$.

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Translated by N.H.C.